Problem 5089. In $\triangle ABC$ let $AB = c$, $BC = a$, $CA = b$, $r =$ the in-radius and R_a, r_b and $r_c =$ the es-radii, respectively. Prove or disprove that

$$\frac{(r_a - r)(r_b + r_c)}{r_a r_c + r r_b} + \frac{(r_c - r)(r_a + r_b)}{r_c r_b + r r_a} + \frac{(r_b - r)(r_c + r_a)}{r_b r_a + r r_c} \geq 2\left(\frac{ab}{b^2 + ca} + \frac{bc}{c^2 + ab} + \frac{ca}{a^2 + bc}\right)$$

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Let Δ and s be the area and the semiperimeter of $\triangle ABC$, respectively. By the well known trigonometrical identities: $r = \frac{\Delta}{s}$, $r_a = \frac{\Delta}{s-a}$, $r_b = \frac{\Delta}{s-b}$ and $r_c = \frac{\Delta}{s-c}$ we have

$$\begin{aligned}
\frac{(r_a - r)(r_b + r_c)}{r_a r_c + r r_b} &= \frac{\left(\frac{\Delta}{s-a} - \frac{\Delta}{s}\right)\left(\frac{\Delta}{s-b} - \frac{\Delta}{s-c}\right)}{\frac{\Delta^2}{(s-a)(s-c)} + \frac{\Delta^2}{s(s-b)}} = \\
&= \frac{(s - s + a)(s - c + s - b)}{s(s-b) + (s-a)(s-c)} = \\
&= \frac{a^2}{s^2 - sb + s^2 - sc - sa + ac} = \\
&= \frac{a^2}{2s^2 - s(a+b+c) + ac} = \\
&= \frac{a^2}{2s^2 - 2s^2 + ac} = \frac{a}{c}
\end{aligned}$$

and similar inequalities hold for cyclic permutations of a, b, c. Therefore, our inequality is equivalent to

$$\frac{a}{c} + \frac{c}{b} + \frac{b}{a} \geq \frac{2ab}{b^2 + ca} + \frac{2bc}{c^2 + ab} + \frac{2ca}{a^2 + bc} \qquad (*)$$

According to **AM-HM** inequality we get

$$\frac{\frac{b}{a} + \frac{c}{b}}{2} \geq \frac{2}{\frac{b}{a} + \frac{c}{b}} = \frac{2ab}{b^2 + ca}$$

Building up two similar inequalities and adding up all of them, we get (*) and the conclusion follows. Equality holds for $a = b = c$. □